The authors in Refs. 2 and 4 should not compare their exact solutions with the Zuravski formula, since this formula is not applicable for beams of variable cross section. The present author shows that a comparison of the solutions in Refs. 2 and 4 with those from Eq. (1) results in satisfactory agreement. These solutions could also be compared using the method derived in Ref. 5 for beams of variable height and constant thickness.

The tapered beam of constant thickness shown in Fig. 3 will now be analyzed for shearing stress using theory of elasticity and strength of materials, Eq. (1). An elasticity solution to the beam problem of Fig. 3 may be obtained by a proper superposition of the solutions given for the beams of Figs. 64 and 65 in Ref. 2. Such a solution is

$$\tau_{xy}=\sigma_r \sin 2\theta/2+\tau_{r\theta}\cos 2\theta \quad \text{where } \sigma_\theta=0 \tag{3a}$$

$$\sigma_r=-\frac{2P\sin\theta}{r(2\alpha-\sin 2\alpha)}+\frac{2Pa\sin 2\theta}{r^2(\sin 2\alpha-2\alpha\cos 2\alpha)}$$

$$\tau_{r\theta}=-\frac{Pa(\cos 2\theta-\cos 2\alpha)}{r^2(\sin 2\alpha-2\alpha\cos 2\alpha)} \tag{3b}$$

A narrow wedge ($\alpha=3$) and a wide wedge ($\alpha=20$) of the tapered beam shown in Fig. 3 have been solved for the shearing stress at several points. The results calculated by theory of elasticity Eq. (3) and strength of materials Eq. (1) are summarized in Table 2. The maximum shearing stress, which occurs at point A (Fig. 3) is also included. These results again indicate that Eq. (1) may be used for the narrow tapered beam. Since solutions obtained from Eq. (1) may be used for tapered beams with small wedge angles, it appears that Eq. (1) may also be used for beams of various other shapes and thickness.

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Peak Distributions of Random Response Processes

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It has been recognized[1] that, from the standpoint of the strength of a linear or nonlinear structure, it is more important to study the stochastic distribution of the peak and the highest peak of the response, rather than that of the response itself, when systems are analyzed against random excitations of the stationary or nonstationary type.

A peak or a maximum in a sample random function $x(t)$ of a continuously valued random process if also continuous with time t and occurs when $\dot{x}(t)$ is zero and $\ddot{x}(t)$ is negative, where $x(t)$ represents the response of the system, single or multidegree of freedom, maybe stress, displacement, or strain at a critical point or zone and $\dot{x}(t)$, $\ddot{x}(t)$ its first and second derivatives, respectively.

In a large class of problems, the designer will only be interested in estimating the distribution of the largest of the maximum occurring within a specified period, and the distribution of the peaks may be of little concern to him.

Standard methods to estimate the distribution of the peaks making use of the joint probability density function of $x(t)$, $\dot{x}(t)$, and $\ddot{x}(t)$ involve tedious numerical calculations.[2] However, no exact procedure is available for the evaluation of the distribution of the largest of the peaks. An approximate solution to these problems is attempted in this study.

Analysis

If $x(t)$ represents a nonstationary random process, say the response of a single degree of freedom system, the number of extrema in $x(t)$, $\epsilon(\lambda, t_1, t_2)$ above a specified level λ within a time interval (t_1, t_2) can be expressed as[2]

$$\epsilon(\lambda,t_1,t_2)=\int_{t1}^{t2}|\ddot{x}(t)|\delta[\dot{x}(t)]\mathbf{1}[x(t)-\lambda]\,\mathrm{d}t \tag{1}$$

where $\mathbf{1}$ [ ] represents Heaviside's step function and δ [ ] Dirac's delta function.

Following Rice,[3] the expected number of times a level is crossed from below in an interval (t_1, t_2) is given by

$$E[N(\lambda,t_1,t_2)]=\int_{t1}^{t2}\int_0^\alpha|\dot{x}|p(\lambda,\dot{x};t)\,\mathrm{d}\dot{x}\mathrm{d}t \tag{2}$$

where $p(x, \dot{x}; t)$ is the joint function of $x(t)$ and $\dot{x}(t)$ given by

$$p(x,\dot{x},t)=\frac{1}{2\pi\sigma_1\sigma_2(1-p^2)^{1/2}}\exp\left[-\frac{1}{2(1-p^2)}\left\{\left[\frac{x}{\sigma_1}\right]^2-\frac{2px\dot{x}}{\sigma_1\sigma_2}+\left[\frac{\dot{x}}{\sigma_2}\right]^2\right\}\right] \tag{3}$$

where $p(t)$ is the correlation function of $x(t)$ and $\dot{x}(t)$; $\sigma_1(t)$ and $\sigma_2(t)$ are the standard deviations of $x(t)$ and $\dot{x}(t)$, respectively. The expected number of peaks above the zero level of response can then be estimated using Eq. (2) with $\lambda=0$, assuming that there is only one peak associated with the response process crossing this level. The probability distribution function of the peaks at the level λ, F_p (λ, t) is then

$$F_p(\lambda,t)=\frac{\int_{t1}^{t2}\int_0^\alpha|\dot{x}|p(\lambda,\dot{x};t)\mathrm{d}\dot{x}\mathrm{d}t}{\int_{t1}^{t2}\int_0^\alpha|\dot{x}|p(0,\dot{x};t)\,\mathrm{d}\dot{x}\mathrm{d}t} \tag{4}$$

For estimating the probability density and distribution of the largest of the maxima, the results obtained by Davenport[4] are used. The probability $p_L(\lambda,t)$ that the largest of the peaks has a value λ is that one of the maxima has this value and the rest are smaller within the interval considered. The required probability can thus be expressed as:

$$p_L(\lambda,t)=N[1-F_p(\lambda,t)]^{N-1}\mathrm{d}F_p(\lambda,t)/\mathrm{d}\lambda \tag{5}$$

Where N is the expected number of peaks in the interval (t_1, t_2) above the zero level and is given by Eq. (2) with $\lambda=0$. The statistical properties of the distribution can be easily estimated from Eq. (5). For example, the mean of the largest of the maxima, L (λ,t) is given by

$$L(\lambda,t)=\int_0^\alpha p_L(\lambda,t)\lambda\,\mathrm{d}\lambda \tag{6}$$

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Conclusion

Compared to the rigorous procedures[2] the solution to the previously stated problem, given by Eqs. (4) and (5) is approximate, but avoids the cumbersome calculations involved in the former. In this connection, the stochastic analysis of a single degree of freedom system subjected to random wind and seismic excitations to study the response characteristics was undertaken by the authors. The exciting force was assumed to be nonstationary in character, and was represented by the product of a deterministic shape function and a stationary random process characterized by its power spectral density. The choice of deterministic function and power spectral density was based on certain characteristics observed in a large number of past records of excitation process. The application of Eqs. (4) and (5) to study the peak response characteristics of the system revealed that the probability estimates for various appropriate values of λ are about 0.5% below those obtained by an exact procedure.

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Finite Difference Method for Computing Sound Propagation in Nonuniform Ducts

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Introduction

In contrast to the analytic methods which expand solutions in a finite series of modes, the method described in this paper is that of finite differences. Numerical solutions for duct acoustics problems in two-dimensional rectangular ducts of constant cross-section have been obtained by Baumeister[1] for the case of no flow and by Baumeister and Rice[2] in the flow case. For ducts with variable cross-section, Alfredson[3] has used a stepped duct approach while Eversman et al.[4] have used the method of weighted residuals (Galerkin's Method.) A finite difference solution for a duct with variable cross-section will now be described.

Governing Equations and Boundary Conditions

For a three-dimensional axially symmetric duct with no flow and steady state, the equations of continuity and momentum may be combined to yield[1,2,7]

$$\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \left(\frac{\omega}{c}\right)^2 p = 0 \tag{1}$$

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where z is the axial coordinate, r is the radial coordinate, and time dependence has been assumed to be of the form $e^{i\omega t}$. In these coordinates, the following boundary conditions are prescribed

$$p(z,r) = f(z,r) \quad \text{at the entrance}$$

$$(\partial p/\partial n) + i\omega/(c\zeta_r)p = 0 \quad \text{on the lateral boundaries}$$

$$(\partial p/\partial n) + i\omega/(c\zeta_z)p = 0 \quad \text{at the exit,} \tag{2}$$

where $(\partial/\partial n)$ is the directional derivative in the direction of the outer normal to the boundary, and ζ_r is the specific acoustic impedence of the wall lining, while ζ_z is the specific acoustic impedence of the exit.

Method of Solution

The well-known Riemann Mapping Theorem establishes that an arbitrary simply connected domain in the plane can be mapped conformally onto an open rectangle with vertices (0,1), (0,-1), $(X,-1)$, and $(X,1)$. (For a diagram of such a map see Fig. 1.) If the map is of the form

$$z = z(x,y); \; r = r(x,y)$$

then for it to be conformal, the Cauchy-Riemann equations:

$$(\partial z/\partial x) = (\partial r/\partial y), \qquad (\partial z/\partial y) = -(\partial r/\partial x)$$

must be satisfied. In transformed coordinates Eq. (1) becomes

$$\begin{aligned} 0 = {} & \frac{\partial^2 \hat{p}}{\partial x^2} + \frac{\partial^2 \hat{p}}{\partial y^2} - \frac{1}{r(x,y)}\frac{\partial z(x,y)}{\partial y}\frac{\partial \hat{p}}{\partial x} \\ & + \frac{1}{r(x,y)}\frac{\partial z(x,y)}{\partial x}\frac{\partial \hat{p}}{\partial y} \\ & + (2\pi\eta)^2\left\{\left[\frac{\partial z(x,y)}{\partial x}\right]^2 + \left[\frac{\partial z(x,y)}{\partial y}\right]^2\right\}\hat{p} \end{aligned} \tag{3}$$

where $\hat{p}(x,y) = p(z,r)$ and $\eta = (1/2\pi)(\omega/c)$ is the dimensionless frequency. The boundary conditions (2) become

$$\hat{p}(0,y) = f(z(0,y), r(0,y)) = g(y) \quad \text{at the entrance}$$

$$\frac{\partial \hat{p}}{\partial y} = 0 \qquad \text{at } y=0$$

$$\frac{\partial \hat{p}}{\partial y} = -2\pi\eta i p\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} / \zeta_y \quad \text{at } y=1$$

$$\frac{\partial \hat{p}}{\partial x} = -2\pi\eta i p\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} / \zeta_x \quad \text{at } x=X \tag{4}$$

To obtain a solution of Eqs. (3) and (4) on a cylinder, the following finite difference scheme is employed.[1,2,6]

Set

$\delta x = X/(n+1)$,

$\delta y = 1/(m+1)$,

$x_j = j\,\delta x, \qquad j = 1,\dots,n$

$y_k = k\,\delta y, \qquad k = 1,\dots,m$

$z_{j,k} = \partial z(x_j, y_k)/\partial y$

$\hat{z}_{j,k} = \partial z(x_j, y_k)/\partial x, \qquad j = 1,\dots,n$

$r_{j,k} = r(x_j, y_k), \qquad k = 1,\dots,m$